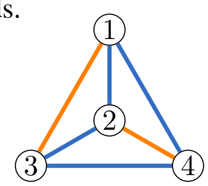
1. There are 4 cities and exactly one road between every pair of cities. How many ways can you paint the roads orange and blue such that it is possible to travel between any two cities on only orange roads, and it is also possible to travel between any two cities on only blue roads? Prove your answer.

   For example, the following coloring of the roads would *not* be allowed, because there is no way to travel between 1 and 2 on orange roads.

2. Quinn places a queen on an empty 8×8 chessboard. She keeps the board secret, but she tells Alex the row that the queen is in, and she tells Adrian the column. Then she asks Alex and Adrian, alternately, whether or not they know how many moves are available to the queen.

   Alex says, "I don't know."

   Adrian says, "I didn't know before, but now I know."

   Alex says, "Now I know, too."

   How many moves must be available to the queen? (Note: A queen can move to any square in the same row, column, or diagonal on the chessboard, except for its current square.)

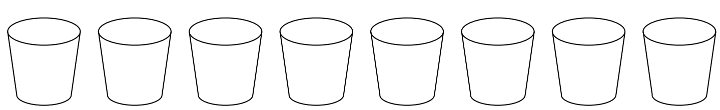
3. Three distinct points A, B, and C lie on a circle with center O, such that $\angle AOB$ and $\angle BOC$ each measure 60 degrees. Point P lies on chord $\overline{AC}$ and the triangle BPC is drawn. If P is chosen so that it maximizes the length of the altitude from C to $\overline{BP}$, then determine the ratio $AP : PC$.

4. The polynomial $P(x)$ has integer coefficients, and $n\,P(n) \equiv 1 \pmod{16}$ for every odd positive integer n. That is, $nP(n)$ is 1 greater than a multiple of 16. Find, with proof, the minimum possible degree of $P(x)$.

5. The Great Pumpkin challenges you to the following game. In front of you are 8 empty buckets. Each bucket is able to hold 2 liters of water. You and the Great Pumpkin take turns, with you going first.

   On your turn, you take 1 liter of water and distribute it among the buckets in any amounts you like. On the Great Pumpkin's turn, it empties two of the buckets of its choice. The Great Pumpkin is defeated if you cause any of the buckets to overflow.

   Given sufficiently many turns, can you defeat the Great Pumpkin no matter how it plays? Prove it, or prove that it is impossible.

6. In a deck of n cards, there is one card of each number from 1 to n. Let a_n be the number of orderings of the deck such that the first card is less than the second card, the second card is greater than the third card, the third card is less than the fourth card, and so on. For example, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, and $a_4 = 5$. Determine, for all n, the remainder when a_n is divided by 4.